PARTIAL DIFFERENTIAL EQUATION WITH VARIABLE CO-EFFICIENTS :-

$$Rr + Ss + Tt + Pp + Qq + Zz = W \quad (1)$$

is the general form.

$$\eta = \eta(x, y)$$

$$\zeta = \zeta(x, y)$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial z}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial z}{\partial \zeta}\frac{\partial \zeta}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x} p$$

$$= \frac{\partial p}{\partial x}$$

$$= \frac{\partial^2 z}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial \zeta \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \zeta}{\partial y}$$

$$+ \frac{\partial^2 z}{\partial \zeta^2}\left(\frac{\partial \zeta}{\partial x}\right)^2 + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial z}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial^2 z}{\partial \zeta \partial \eta}\left(\frac{\partial \eta}{\partial x}\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial x}\frac{\partial \eta}{\partial y}\right)$$

$$+ \frac{\partial^2 z}{\partial \zeta^2}\frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial y \partial x} + \frac{\partial z}{\partial \zeta}\frac{\partial^2 z}{\partial y \partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}$$

$$= \frac{\partial^2 z}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial \eta \partial \zeta}\frac{\partial \eta}{\partial y}\frac{\partial \zeta}{\partial y}$$

$$+ \frac{\partial^2 z}{\partial \zeta^2}\left(\frac{\partial \zeta}{\partial y}\right)^2 + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial z}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2}$$

Ex-19

Reduce the Eqn to canonical form

$$xyr + (x^2-y^2)s - xyt + py - qy$$
$$= 2(x^2-y^2) \quad \text{---} (1)$$

Solution

given

$$xyr - (x^2-y^2)s - xyt + py - qy = 2(x^2-y^2)$$

Here

$$R = xy$$
$$S = -(x^2-y^2)$$
$$T = -xy$$

Now put these value in

$$R\alpha^2 + S\alpha + T = 0$$

$$xy\alpha^2 + \{-(x^2-y^2)\}\alpha + (-xy) = 0$$

$$\Rightarrow xy\alpha^2 - (x^2-y^2)\alpha - xy = 0$$

$\Rightarrow xy\alpha^2 - x^2\alpha + y^2\alpha - xy = 0$

$\Rightarrow x\alpha(y\alpha - x) + y(y\alpha - x) = 0$

$\Rightarrow (y\alpha - x)(x\alpha + y) = 0$

$y\alpha - x = 0$

$y\alpha = x$

$\alpha = x/y$

$x\alpha + y = 0$

$x\alpha = -y$

$\alpha = -y/x$

NOW

$$\frac{dy}{dx} + \alpha_1(x,y) = 0$$

$$\frac{dy}{dx} + \alpha_2(x,y) = 0$$

$$\Rightarrow \frac{dy}{dx} - y/x = 0$$

$$\frac{dy}{dx} + x/y = 0$$

$$\Rightarrow \frac{dy}{dx} - y/x = 0$$

$$\frac{dy}{dx} = y/x$$

$$\frac{dy}{y} = \frac{dx}{x}$$

integrating

$$\log x = \log y$$

$$\log y - \log x = \log \gamma$$

$$\log y/x = \log \gamma$$

$$y/x = \gamma \quad \text{———} \quad (2)$$

$$\frac{dy}{dx} + x/y = 0$$

$$\frac{dy}{dx} = -x/y$$

$$y\,dy = -x\,dx$$

$$x\,dx + y\,dy = 0$$

integrating

$$x^2/_2 + \frac{y^2}{2} = \tau/_2$$

$$x^2 + y^2 = \tau \quad \text{———} \quad (3)$$

day

Now

$$p = \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial z}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

$$= \frac{\partial z}{\partial \eta}\left(-y/x^2\right) + \frac{\partial z}{\partial \zeta}\cdot 2x$$

$$q = \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial z}{\partial \zeta}\frac{\partial \zeta}{\partial y}$$

$$= \frac{1}{x}\frac{\partial z}{\partial \eta} + \frac{\partial z}{\partial \zeta}\,2y$$

$$r = \frac{\partial^2 z}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial \zeta \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \zeta}{\partial x}$$

$$+ \frac{\partial^2 z}{\partial \zeta^2}\left(\frac{\partial \zeta}{\partial x}\right)^2 + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial z}{\partial \zeta}\frac{\partial^2 \zeta}{\partial x^2}$$

$$= \left(-y/x^2\right)^2\frac{\partial^2 z}{\partial \eta^2} + 2\,(2x)\cdot\left(-y/x\right)\frac{\partial^2 z}{\partial \zeta \partial \eta}$$

$$+ 4x^2\frac{\partial^2 z}{\partial \zeta^2} + \frac{2y}{x^3}\frac{\partial z}{\partial \eta} + 2\frac{\partial z}{\partial \zeta}$$

$$s = \frac{\partial^2 z}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial^2 z}{\partial \zeta \partial \eta}\left(\frac{\partial \eta}{\partial x}\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial x}\frac{\partial \eta}{\partial y}\right)$$

$$+ \frac{\partial^2 z}{\partial \zeta^2}\cdot\frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial y \partial x} + \frac{\partial z}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y \partial x}$$

$$= \left(-\frac{y}{x^2}\right)\frac{1}{x}\frac{\partial^2 z}{\partial \eta^2} + \left\{2y\left(-\frac{y}{x^2}\right) + \left(-\frac{y}{x^2}\right)\right.$$

$$+\, 2x\left\{\frac{\partial^2 z}{\partial \zeta \partial \eta} + 4xy\frac{\partial^2 z}{\partial \zeta^2} - \frac{1}{x^2}\frac{\partial z}{\partial \eta}\right.$$

$$t = \frac{\partial^2 z}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial \eta \partial \zeta}\frac{\partial \eta}{\partial y}\frac{\partial \zeta}{\partial y}$$

$$+ \frac{\partial^2 z}{\partial \zeta^2}\left(\frac{\partial \zeta}{\partial y}\right)^2 + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial z}{\partial \zeta}\frac{\partial^2 \zeta}{\partial y^2}$$

$$= \left(\frac{1}{x}\right)^2\frac{\partial^2 z}{\partial \eta^2} + 2\frac{1}{x}(2y)\frac{\partial^2 z}{\partial \eta \partial \zeta}$$

$$+ 4y^2\frac{\partial^2 z}{\partial \zeta^2} + 2\frac{\partial z}{\partial \zeta}$$

putting ~~the~~ value of p, q, r, s, and t in eqn (1) we get

$$-\frac{2}{x^2}(x^2+y^2)\frac{\partial^2 z}{\partial \eta \partial \zeta} = 2(x^2-y^2)$$

$$\Rightarrow \frac{\partial^2 z}{\partial \eta \partial \zeta} = \frac{(y^2-x^2)\,x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial \eta \partial z} = \frac{\eta^2 - 1}{(\eta^2+1)^2}.$$

Required canonical form.

NOTE

$$R\alpha^2 + S\alpha + T = 0$$

Case - I

$S^2 - 4RT > 0$ The roots are distinct.

$$\frac{dy}{dx} + \alpha_1(x,y) = 0$$

$$\eta = f_1(x,y)$$

$$\frac{dy}{dx} + \alpha_2(x,y) = 0$$

$$z = f_2(x,y)$$

It is Hyperbolic

Case - II

$S^2 - 4RT = 0$, The roots are equal.

It is parabolic.

Case - III

$S^2 - 4RT < 0$, The roots are complex

$$u = \frac{1}{2}(\eta + z)$$

$$v = \frac{1}{2} i (z - \eta)$$

It is Elliptic.

so

$$\frac{\partial z}{\partial \eta} = \frac{1}{2}\left(\frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial z}{\partial \xi} = \frac{1}{2}\left(\frac{\partial z}{\partial u} + i\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{1}{4}\left\{\left(\frac{\partial^2 z}{\partial u^2} + i\frac{\partial^2 z}{\partial v \partial u}\right) - \frac{i}{4}\left(\frac{\partial^2 z}{\partial u \partial v} + i\frac{\partial^2 z}{\partial v^2}\right)\right.$$

$$= \frac{1}{4}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Ex-20

Reduce the Eqn

$xy^3 r - 2x^2y^2 s + yx^3 t - y^3 p - x^3 q = 0$

to canonical form.

Soln Given

$xy^3 r - 2x^2y^2 s + yx^3 t - y^3 p - x^3 q = 0$

Here

$R = xy^3$, $S = -2x^2y^2$, $T = yx^3$

Now

$S^2 - 4RT = (-2x^2y^2)^2 - 4 \cdot xy^3 \cdot yx^3$

$= 4x^4y^4 - 4x^4y^4$

$= 0$

$S^2 - 4RT = 0$

So

$$xy^3\alpha^2 - 2x^2y^2\alpha + yx^3 = 0$$

$$\Rightarrow xy(y^2\alpha^2 - 2xy\alpha + x^2) = 0$$

$$\Rightarrow (y\alpha)^2 - 2\cdot y\alpha\cdot x + (x)^2 = 0$$

$$\Rightarrow (y\alpha - x)^2 = 0$$

$$\Rightarrow (y\alpha - x)(y\alpha - x) = 0$$

$$\Rightarrow y\alpha - x = 0$$

$$\Rightarrow y\alpha = x$$

$$\Rightarrow \alpha = x/y$$

$$\alpha = \alpha_1 = \alpha_2 = x/y$$

Now

$$\frac{dy}{dx} + \alpha_1(x,y) = 0$$

$$\frac{dy}{dx} + x/y = 0$$

$$\frac{dy}{dx} = -x/y$$

$$\Rightarrow y\,dy = -x\,dx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + \eta/2$$

$$\Rightarrow \eta = x^2 + y^2$$

we take $\zeta = x^2 - y^2$

$$p = 2x\frac{\partial z}{\partial \eta} + 2x\frac{\partial z}{\partial \zeta}$$

$$q = 2y\frac{\partial z}{\partial \eta} - 2y\frac{\partial z}{\partial \zeta}$$

$$r = 4x^2\frac{\partial^2 z}{\partial \eta^2} + 8x^2\frac{\partial^2 z}{\partial \eta\,\partial \zeta} + 4x^2\frac{\partial^2 z}{\partial \zeta^2} + 2\frac{\partial z}{\partial \eta} + 2\frac{\partial z}{\partial \zeta}$$

$$s = 4xy\frac{\partial^2 z}{\partial \eta^2} - 4xy\frac{\partial^2 z}{\partial \zeta^2}$$

$$t = 4y^2\frac{\partial^2 z}{\partial \eta^2} - 8y^2\frac{\partial^2 z}{\partial \zeta\,\partial \eta} + 4y^2\frac{\partial^2 z}{\partial \zeta^2} + 2\frac{\partial z}{\partial \eta} - 2\frac{\partial z}{\partial \zeta}$$

substituting these value in given Eqn, we get

$$\frac{\partial^2 z}{\partial \zeta^2} = 0$$

~~which is desired canonical form.~~

which is desired canonical form.

Ex - 21

Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form.

Solution

Given

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Here

$$R = 1, \quad S = 0, \quad T = x^2$$

Now

$$S^2 - 4RT = 0^2 - 4.1.x^2$$

$$= -4x^2 < 0$$

Hence

$$R\alpha^2 + S\alpha + T = 0$$

$$\alpha^2 + 0.\alpha + x^2 = 0$$

$$\alpha^2 + x^2 = 0$$

$$\alpha^2 = -x^2$$

$$\alpha = \sqrt{-x^2}$$

$$\alpha = \pm ix$$

$$\alpha_1 = ix$$

$$\alpha_2 = -ix$$

$$\frac{dy}{dx} + ix = 0$$

$$\frac{dy}{dx} + ix = 0$$

$$\frac{dy}{dx} = -ix$$

$$dy = -ix\,dx$$

$$iy = -x^2/2 + \eta$$

$$\eta = iy + x^2/2 = x^2/2 + iy$$

$$z = -iy + x^2/2 = x^2/2 - iy$$

Hence

$$u = x^2/2, \quad v = y$$

$$\eta = iv + u$$

$$z = -iv + u$$

$$\frac{\partial z}{\partial \eta} = \frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial z} = \frac{\partial z}{\partial u} + i\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial z \partial \eta} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

$$R' = R\left(\frac{\partial \eta}{\partial x}\right)^2 + S\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + T\left(\frac{\partial \eta}{\partial y}\right)^2$$

$$S' = R\frac{\partial \eta}{\partial x}\frac{\partial z}{\partial x} + \frac{1}{2}S\left(\frac{\partial \eta}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial \eta}{\partial y}\frac{\partial z}{\partial x}\right) + T\frac{\partial \eta}{\partial y}\frac{\partial z}{\partial y}$$

$$T' = R\left(\frac{\partial z}{\partial x}\right)^2 + S\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + T\left(\frac{\partial z}{\partial y}\right)^2$$

$$p' = R\frac{\partial^2 \eta}{\partial x^2} + S\frac{\partial^2 \eta}{\partial y \partial x} + T\frac{\partial^2 \eta}{\partial y^2} + P\frac{\partial \eta}{\partial x} + Q\frac{\partial \eta}{\partial y}$$

$$Q' = R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial y \partial x} + T\frac{\partial^2 z}{\partial y^2} + P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y}$$

Substituting these value in given Eqn

$$2s'\frac{\partial^2 Z}{\partial z \partial \eta} + p'\frac{\partial Z}{\partial \eta} + Q'\frac{\partial Z}{\partial z} = 0$$

$$\frac{\partial Z}{\partial \eta} = \frac{1}{2}\left(\frac{\partial Z}{\partial u} - i\frac{\partial Z}{\partial v}\right)$$

$$\frac{\partial Z}{\partial z} = \frac{1}{2}\left(\frac{\partial Z}{\partial u} + i\frac{\partial Z}{\partial v}\right)$$

$$\frac{\partial^2 Z}{\partial z \partial \eta} = \frac{1}{4}\left(\frac{\partial^2 Z}{\partial u^2} + i\frac{\partial^2 Z}{\partial v \partial u}\right) - \frac{i}{4}\left(\frac{\partial^2 Z}{\partial u \partial v} + i\frac{\partial^2 Z}{\partial v^2}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2 Z}{\partial u^2} + \frac{\partial^2 Z}{\partial v^2}\right)$$

$$2 \cdot 4u \cdot \frac{1}{4}\left(\frac{\partial^2 Z}{\partial u^2} + \frac{\partial^2 Z}{\partial v^2}\right) + 1 \cdot \frac{1}{2}\left(\frac{\partial Z}{\partial u} - i\frac{\partial Z}{\partial v}\right)$$

$$+ 1 \cdot \frac{1}{2}\left(\frac{\partial Z}{\partial u} + i\frac{\partial Z}{\partial v}\right) = 0$$

$$2u\left(\frac{\partial^2 Z}{\partial u^2} + \frac{\partial^2 Z}{\partial v^2}\right) + \frac{1}{2}\frac{\partial Z}{\partial u} - i/2\,\frac{\partial Z}{\partial v} + \frac{1}{2}\frac{\partial Z}{\partial u} + i/2\,\frac{\partial Z}{\partial v} = 0$$

$$2u\left(\frac{\partial^2 Z}{\partial u^2} + \frac{\partial^2 Z}{\partial v^2}\right) + \frac{1}{2}\frac{\partial Z}{\partial u} + \frac{1}{2}\frac{\partial Z}{\partial u} = 0$$

$$\Rightarrow 2u\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = -\frac{1}{2u}\frac{\partial z}{\partial u}$$

which is desired canonical form

SOME STANDARD FORMS OF VARIABLE COEFFICIENTS:

FORM - I

$$r = f_1(x,y)$$
$$s = f_2(x,y)$$
$$t = f_3(x,y)$$

They can be solved by direct integration.

Ex-22 solve
$$s = 4xe^{2y}$$

soln:-

we rewrite the eqn in the form

$$\frac{\partial^2 z}{\partial x \partial y} = 4xe^{2y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 4xe^{2y}$$

$$\partial\left(\frac{\partial z}{\partial y}\right) = 4xe^{2y}\,\partial x$$

Integrating

$$\frac{\partial z}{\partial y} = 2x^2e^{2y} + \phi_1(y)$$

$$\partial z = \{2x^2e^{2y} + \phi_1(y)\}\,\partial y$$

Again integrating

$$z = x^2e^{2y} + \phi_1(y) + \psi_1(x)$$

where ϕ_1 and ψ_1 are arbitary functions.

FORM - II

$$Rr + Pp = W(x,y)$$
$$Ss + Pp = W(x,y)$$
$$Ss + Qq = W(x,y)$$
$$Tt + Qq = W(x,y)$$

They can be solved by method of solving Ordinary linear differential eqn of first order.

EX - 23

$$xr + 2p = 2y$$
$$r + \frac{2}{x}p = 2y/x$$

The Eqn can be written as

$$\frac{\partial p}{\partial x} + \frac{2}{x}p = 2y/x$$

which is a linear Ordinary differential eqn of Order one with p as dependent variable.

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx} = e^{2\ln x}$$
$$= e^{\ln x^2} = x^2$$

$$p \cdot x^2 = x^2 y + \phi_1(y)$$
$$p = y + \frac{\phi_1(y)}{x^2}$$
$$p = y + x^{-2}\phi_1(y)$$
$$\frac{\partial z}{\partial x} = y + x^{-2}\phi_1(y)$$

$$\frac{\partial z}{\partial x} = y + x^{-2}\phi_1(y)$$

$$z = xy - x^{-1}\phi_1(y) + \phi_2(y)$$

ϕ_1 and ϕ_2 are arbitrary function.

FORM – III

$$Rr + Pp + Zz = W(x,y)$$

$$Tt + Qq + Zz = W(x,y)$$

It can be solved by ordinary linear differential Eqn with x or y as independent variable and z as dependent variable.

Ex-24 solve

$$yr + (y^2+1)p + yz = e^x$$

The given Eqn can be written as

$$[yD^2 + (y^2+1)D + y]z = e^x$$

$$D = \frac{\partial}{\partial x}$$

The Auxiliary eqn is

$$yD^2 + (y^2+1)D + y = 0$$

$$\Rightarrow (D+y)(yD+1) = 0$$

The roots of the Auxiliary eqn are

$$-y, -\frac{1}{y}$$

$$C.F = z_c = e^{-xy}\phi_1(y) + e^{-x/y}\phi_2(y)$$

$$P.I = z_p = \frac{1}{(D+y)(yD+1)} e^x$$

$$= \frac{e^{x}}{(1+y)^2}$$

Hence

$$z = \phi_1(y)\, e^{-xy} + \phi_2(y)\, e^{-x/y} + \frac{e^{x}}{(1+y)^2}$$

FORM - IV

$$Rr + Ss + Pp = W$$

$$Ss + Tt + Qq = W$$

It can be solved by Lagrange method

Ex - 25

$$2xr - ys + 2p = 4xy^2$$

The Eqn can be written as

$$2x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y} = 4xy^2 - 2p$$

Lagrange's auxiliary Eqn are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dp}{4xy^2 - 2p}$$

Taking the first two ratio, we get

$$\frac{dx}{2x} = \frac{dy}{-y}$$

$$\frac{dx}{x} = -2\frac{dy}{y}$$

integrating

$$\ln x = -2\ln y + \ln c$$

$$\ln x + \ln y^2 = \ln c$$

$$\ln xy^2 = \ln c$$

$$xy^2 = c$$

Taking first and third ratio, we get

$$\frac{dx}{2x} = \frac{dp}{4xy^2 - 2p}$$

$$\Rightarrow \frac{dp}{dx} = \frac{4xy^2 - 2p}{2x}$$

$$\Rightarrow \frac{dp}{dx} = 2y^2 - p/x$$

$$\Rightarrow \frac{dp}{dx} + p/x = 2y^2$$

$$\Rightarrow \frac{dp}{dx} + p/x = 2c/x$$

which is a linear eqn

$$I.F = e^{\int \frac{dx}{x}} = x$$

$$p \cdot x = 2cx + c_2$$

$$p \cdot x = 2x^2y^2 + c_2$$

$$p = 2xy^2 + \frac{1}{x} c_2$$

Integrating

$$z = x^2y^2 + \int \frac{1}{x} c_2(xy^2)$$

$$z = x^2y^2 + c_2(xy^2) + c_3(y)$$

which is required soln

Derivation of heat Equation

The simple example of the heat equation is flow of heat in a solid bar of conducting material.

To derive the eqn, consider a homogenous bar of uniform cross section α, which is perfectly insulated so the heat flow in the x-direction only.

Let $u(x,t)$ be the temperature at any point of the bar.
Here we assume that the side of the bar are insulated. the loss of heat is negligib due to conduction.
one end of the bar as origin and directi of flow as the positive x-axis.

① The temperature $u(x,t)$ at any point of the bar depends on the distance x of the point from one end at the time 't'

② The temperature at all points of any cross section is same.

Let Δu be the temperature change in a slab of thickness Δx of the bar.

The rate of increase of heat of the slab

$$= s\rho\alpha\,\Delta x\,\frac{\partial u}{\partial t}$$

where s is the specific heat.
ρ is the density of material.

This rate of increase of heat must be equal to the difference of the rates of inflow and outflow of heat through Δx.

The rate at which heat flows through an area is proportional to the area is proporto α of the cross section conductivity k of the material and the temperature gradiant $\frac{\partial u}{\partial x}$ normal to the area.

Now

Rate of inflow heat (Q_1) at a distance $x = -k\alpha\left(\frac{\partial u}{\partial x}\right)_x$ per second

Rate of outflow heat Q_2 at a distance $(x+\Delta x) = -k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$ per second.

$-$ve sign indicate that heat flows from higher to lower temperature.

Hence rate of increase of heat in the slab with thickness Δx is

$$Q_1 - Q_2 = k\alpha\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

Now

$$s\rho\alpha\Delta x\left(\frac{\partial u}{\partial t}\right) = k\alpha\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k}{s\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right]$$

Now taking limit $\Delta x \to 0$ we get

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)$$

where $c^2 = \frac{k}{s\rho}$ (Diffusivity of the substance)

which is required heat eqn of one dimensional heat eqn.

Derivation of wave Equation :-

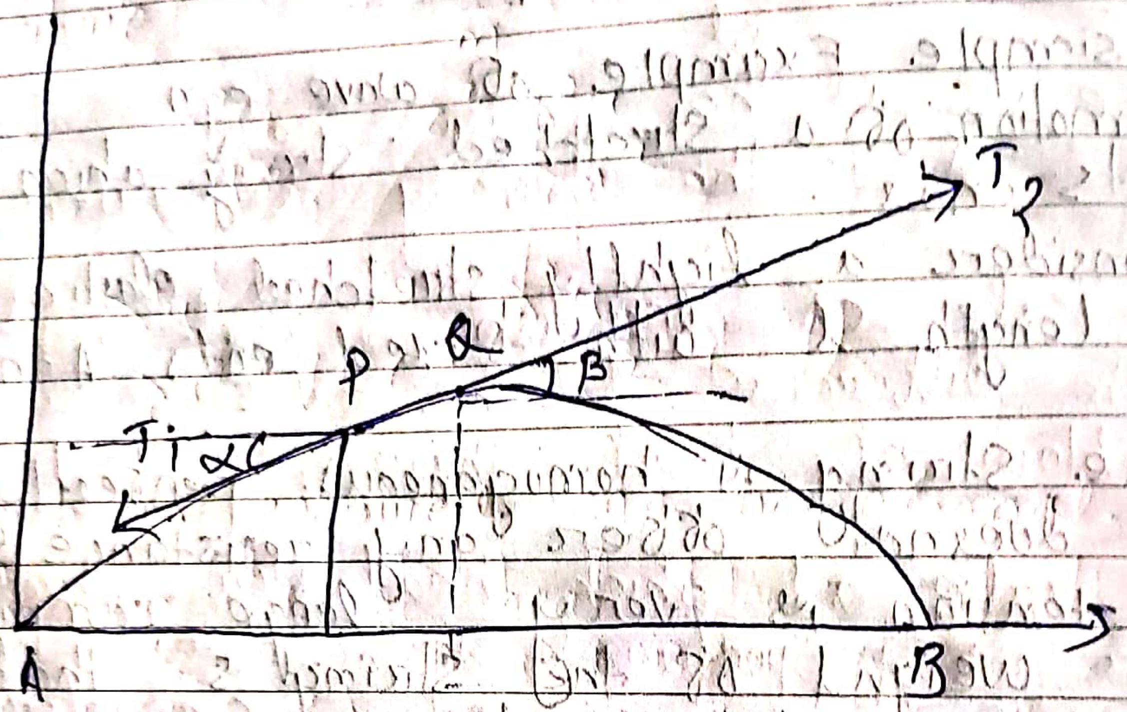
A simple Example of wave eqn
The motion of a stretched string which is fixed at its ends.

consider a lightly stretched elastic string of length l with fixed ends A and B.

① The string is homogeneous perfectly Elastic. and doesnot offer any resistance to bending
② the tension is very large compaired to the weight of the string s. that the effect of the gravitational force on the string can neglected.

③ The motion of the string is a small transverse vibration in a vertical plane. i e Each particle of the string moves strictly vertically

Now
to obtain the differential eqn consider the forces acting on a small portion of the string.
since the string doesnot offer resistance to bending the tension is tangential to the curve of the string at each point.



Let T_1 and T_2 be the tensions at the end points P and Q of the portion. Since there is no motion in horizontal direction.

The Horizontal components of the tension must be constant.

$T_1 \cos\alpha = T_2 \cos\beta = T =$ constant.

The vertical components of T_1 and T_2 are $-T_1 \sin\alpha$ and $T_2 \sin\beta$ respectively.

The upward acceleration is $\frac{\partial^2 u}{\partial t^2}$

If m be the mass per unit length of the

undeflected string and Δx be the length of the portion of the undeflected string Newton's second laws of motion.

$$T_2 \sin\beta - T_1 \sin\alpha = m \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{m \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \tan\beta - \tan\alpha = \frac{m \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

Now

$\tan\alpha$ and $\tan\beta$ are the slope of the curve of the string at x and $x+\Delta x$.

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}.$$

Hence

$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} = \frac{m}{T} \frac{\partial^2 u}{\partial t^2}$$

taking limit $Q \to P$

$$\Delta x \to 0$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c^2 = \frac{T}{m}$

ie $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

which is required one dimensional wave eqn.

The Laplace eqn

consider a flow of heat in a metal plate in the directions of its length (x-axis) and breadth (y-axis) respectively where there is no flow of heat along the direction of the normal (z-axis) to the plane of the rectangle.

The temperature at any point is independent of the z-coordinate and depends on x, y and t

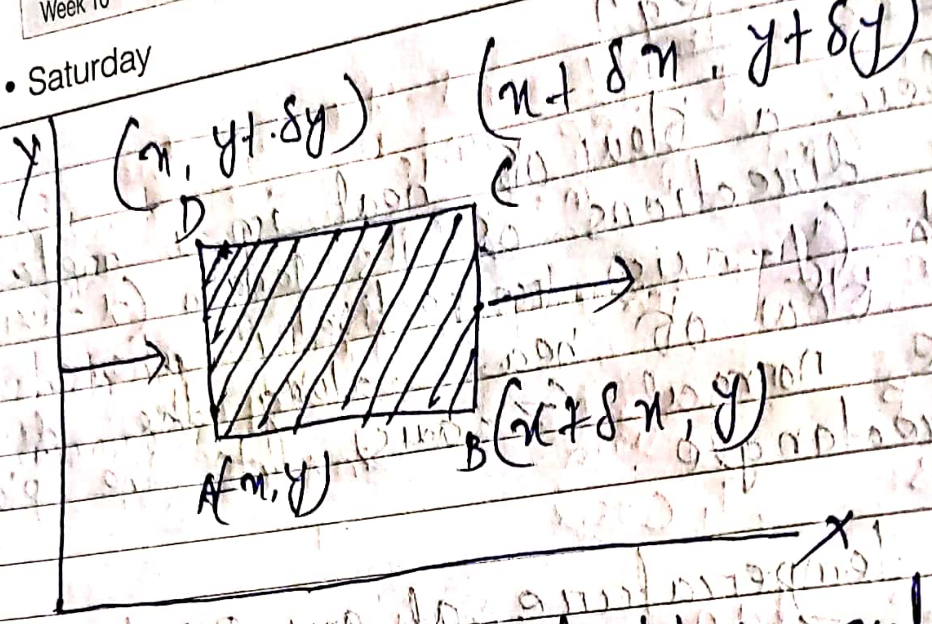
consider the flow of heat in a rectangular plate with side δx and δy.

Let the metal plate be uniform thickness α density ρ, specific heat σ and thermal conductivity k.

Quantity of heat enter the plate per second from the side AB

$$= -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)$$

similarly the quantity of heat that enters the plate per second from the side AD $= -k\alpha\,\delta x\left(\frac{\partial u}{\partial x}\right)$



The quantity of heat flows out through the sides CD and BC per second are $-k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$ and $-k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta y}$

Hence total gain of heat by the rectangular element ABCD per second

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= -k\alpha(\delta x)(\delta y)\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right]$$

The rate of gain of heat

$$= \gamma\rho\,\delta x\,\delta y\left(\frac{\partial u}{\partial t}\right)$$

$$k\alpha(\delta x)(\delta y)\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]$$

$$= \gamma\rho(\delta u)(\delta y)\frac{\partial u}{\partial t}$$

Dividing both side $\alpha(\delta x)(\delta y)$

taking limit $\delta x \to 0$, $\delta y \to 0$

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\gamma\rho}{\alpha}\frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{\partial u}{\partial t}=\left(\frac{k\alpha}{\gamma\rho}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2=\frac{k\alpha}{\gamma\rho}$

The distribution of temperature in the plate, u does not change with t.

so $\frac{\partial u}{\partial t}=0$

30	31	
2	3	4
9	10	11
16	17	18
23	24	25

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is called Laplace eqn. in two dimension.